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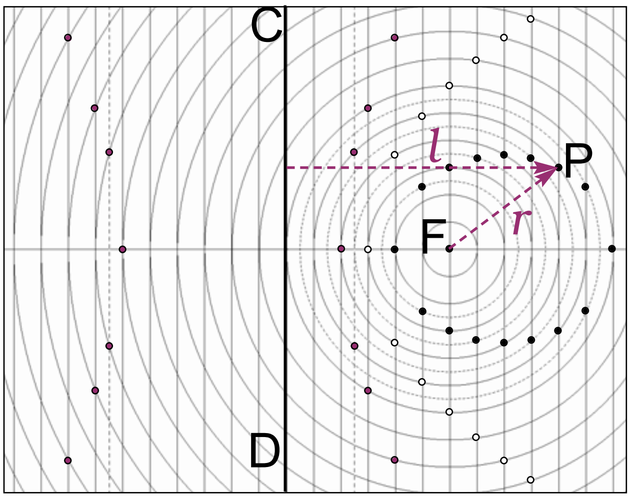
More or less eccentric

Previously in CACTUS we considered simple constructions to study the reflective property of a parabola and of a paraboloid. The purpose was to help middle school students understand the importance of the parabola shape and its ability to focus parallel rays. The examples demonstrated a use of the two *Cabri* products to study a 2D object before considering its 3D equivalent.

This article will consider constructions which are more suitable for senior students. They allow a study of ellipses, parabolas, hyperbolas and their 3D equivalents.

Conics have traditionally been introduced using the specific graph sheet that is shown below. You will find one somewhere in one of the filing cabinets in most schools. They occur in various states of sophistication but the essential features are a set of concentric circles superimposed on a grid of vertical lines. One line, at a multiple of six units from the centre (F) of the circles is darkened and represents the directrix (CD) of some conics, which can be plotted with a focus at F. The sheet is specifically designed to plot conics with eccentricities of 0.5, 1 and 2.



Even in a high-tech classroom, much can be gained by getting students to work through this simple plotting task before opening their computers.

Start by plotting all the points that are equidistant from the directrix and from the focus. If we follow the circle with radius four units until it crosses the line which is four units from the directrix, then the point of intersection will be four units from each. Some time will be saved when the students discover there are two such points.

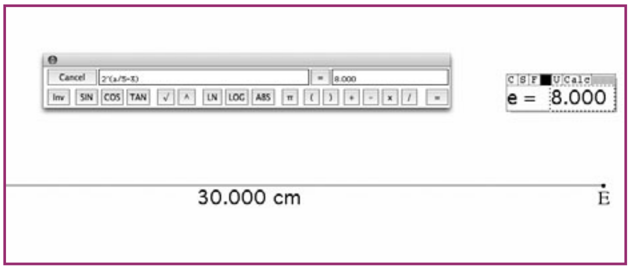
When students have found all the points (they are shown as open circles on the graph) they can assume that there are intermediate values that obey the same rule and draw a smooth curve through the points. Senior school students should quickly recognise this curve as a parabola.

In the same way, students can find points (shown as black dots) that are twice as far from the directrix as they are from the focus. For example, the point P is 10 units from the directrix and 5 units from the focus. We can generalise this to see that for all such points $r = 0.5\ l$. Students can join these points to form an ellipse. We say that this ellipse has an eccentricity of $e = 0.5$.

When students have found the points that are twice as far from the focus as they are from the directrix ($r = 2\ l$) they can join them to form an hyperbola with eccentricity $e = 2$. These points are shown as coloured circles. Students will easily discover the right-hand section of the hyperbola but may need some prompting to find the left-hand section.

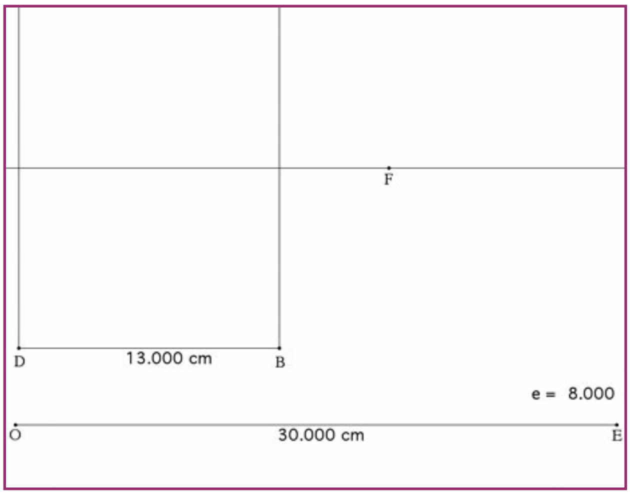
This exercise identifies the variables, introduces eccentricity, and should prepare students to construct a similar model using a dynamic geometry package such as *Cabri Geometry II Plus*.

Students may need some help to insert the variable for eccentricity. In this example I will control *e* using a mouse. To do this I draw a line segment *OE* across the page and measured its length. So that it is easier to check the calculations, I adjust the length of the line to a neat whole number (30 cm). This is much easier after I have reduced the measurement precision in the Preferences.



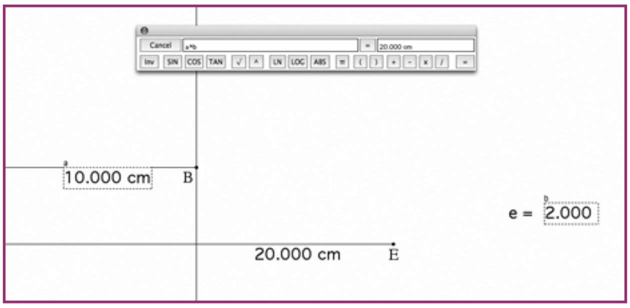
I select the **calculator** and start typing a formula "2^(". Then I click on the measurement of OE: in this case 30.000. A letter "a" appears next to the number and a letter "a" appears in the formula bar. Then I complete the formula 2^(a/5–3). For values of "a" between 0 and 30, (a/5–3) takes values between –3 and +3. I click on the result and pulled it across to the right. After clicking on the result again, I am able to edit the text to read *"e* =". On moving the point E to the left or the right, the value of *e* varies between 0.125 and 8.

Obviously there are other ways to convert a measurement between 0 and 30 to a useful range of eccentricities. This method ensures that we do not have to cope with situations where *e* approaches zero and gives smoother control for the values of *e* between 0.5 and 2.



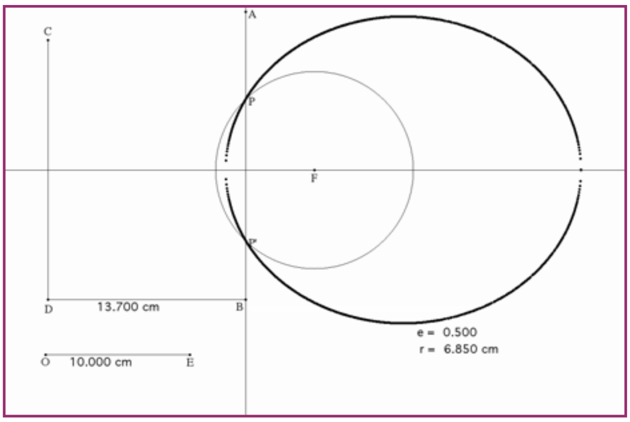
Draw a vertical line segment *CD* and construct its **perpendicular bisector**. Place a point *F* on the perpendicular bisector. Draw a **perpendicular** to *CD* through the point *D* and place a point *B* on the new line. Hide the line *DB* and join *D* and *B* with a line segment. Use **perpendicular** again to draw a vertical line through *B*.

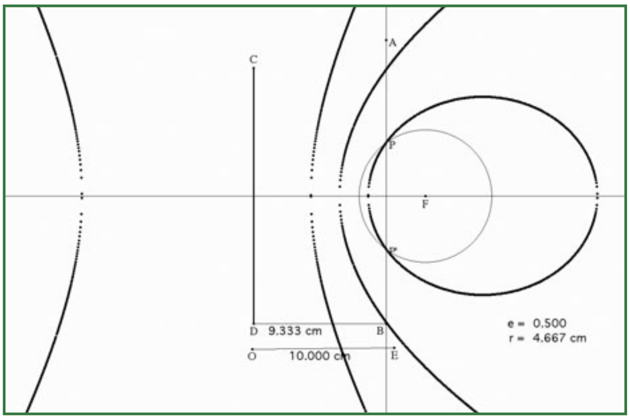
Find the **length** of the segment *DB*. Adjust the position of *B* to give a whole number. If we move the point *E* we can adjust the eccentricity (check the calculation) and the point *B* becomes the handle with which we will drive the finished model.



Select the **calculator** and then click on the measurement of *DB* labelled "a", type "*" to multiply and then click on the value of *e* which is then labelled "b". In the formula window you should see "a*b" and in the result box you have the product of the length DB (*l*) and the eccentricity (*e*). Pull the result to the right, just below the value of *e* and rename the label "result:" as "r =".

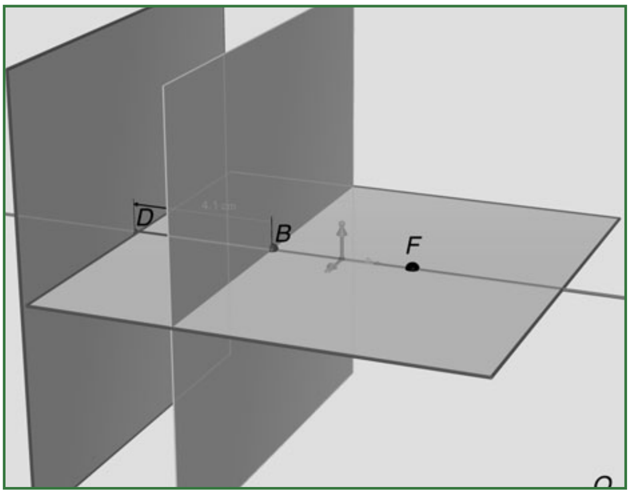
To draw a circle centre *F* and radius *r*, use **compass**, click on the point *F* and then click on the number next to "r =". Find the points *P* and *P′* where the circle cuts the vertical line through *B*. Switch on the **trace** option for *P* and *P′* (the **locus** option does not work very well with this model). Move *E* to choose a value for *e*. Move the point *B* and a conic with $l = |DB|$ and $r = e\,l$ is traced on the screen.





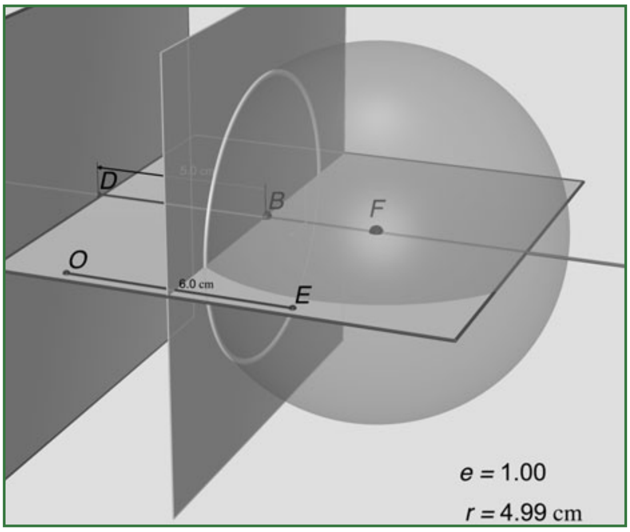
With care it is possible to duplicate the graphing exercise for $e = 0.5$, 1 and 2 so that all of the traces appear on the screen at the same time. This model will also allow students to explore a range of conics with eccentricities ranging between 0.125 and 8.

Cabri 3D can be used in much the same way to build a corresponding 3D model. The opening screen presents a horizontal plane with unit vectors **i**, **j** and **k** at its centre. Place a **line** onto the unit vector **j**. At the left-hand end of the line place a point *D*. At the right-hand end place a point *F*. Between *D* and *F* place a point *B*. Construct the **line segment** *DB* and find its **length** (*l*). Construct planes through the points *D* and *B* **perpendicular** to the line *DF*.



On the front edge of the base plane place a point *O* at a point near the front left-hand corner of the plane. Construct a line through *O* **parallel** to the **j** vector. Hide all of the vectors using control/command-M. Place a point *E* at the right hand end of the new line and measure the **length** of the segment *OE*.

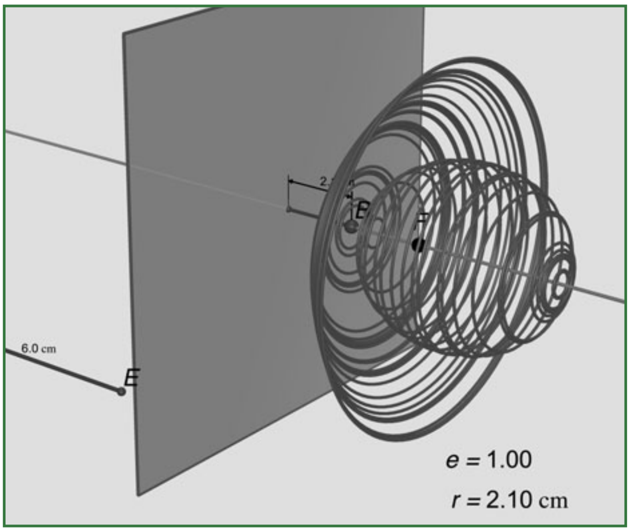
Select the calculator and type "2^(", click on the measurement of *OE* and then complete the formula "2^(a/2–3)". The default scale of *Cabri 3D* needs the modified formula.



Move the result to the screen and call it *e*. Select **calculator** again and multiply the length of *DB* (*l*) by *e* to give the result $r = e\,l$. Move the result to the screen and call it *r*.

Choose **sphere**, point to the centre *F* and the new calculation of *r* for the radius. Move the point *B* until the plane intersects the sphere. Use **intersection curve** to mark the circle of intersection of the plane and the sphere. Increase the thickness of the circle. Hide the base plane, *B* plane and the sphere.

Choose **trajectory** and click on the circle. Now as you move the point *B*, the movement of the circle is sampled along a conic locus.



There is a limit to how many circles can be traced. If you try to trace too many circles, the initial tracing is cannibalised to allow new circles to appear. In this figure I have traced an ellipsoid and a paraboloid. Both sections of an hyperboloid appear on the cover.

This model can be great fun to use and the selection of vivid colours available adds to the enjoyment.